A statistical Theory of Wave-Propagation in Random Medium and the Power Distribution Function-Theory of Cumulants

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1. Introduction

The wave equation considered here is not a particular wave equation but an equation of the form of

$$[L - bq]\,\psi(x) = \eta(x). \tag{1}$$

Here $\psi(x)$ is the wave function of the space coordinates (x) and $\eta(x)$ is the (given) external source; L is an operator operating on ψ, and b is a constant; $q = q(x)$ is a randum function following some statistical distribution. For instance, in the case of an ordinary scalar wave equation,

$$L = -\Delta - k^2, \qquad bq(x) = k^2\Delta\varepsilon(x), \tag{2}$$

where $\Delta\varepsilon$ is the fluctuation part of dielectric constant of the medium.

It is easy to find that all the statistical informations of the wave can be derived from the generalized characteristic function $\langle 0|0\rangle$ defined by

$$\langle 0|0\rangle = \langle \exp[\int dx\{\bar{\eta}(x)\psi(x) + \bar{\eta}^*(x)\psi^*(x)\}]\rangle , \tag{3}$$

where $\langle Q\rangle$ stands for the average value of Q over all possible functions of $q(x)$, and $\bar{\eta}(x)$ and $\bar{\eta}^*(x)$ are arbitrarily given (complex) functions; for instance,

$$\langle\psi(x)\rangle = \{\delta/\delta\bar{\eta}(x)\} \langle 0|0\rangle|_{\bar{\eta}=\bar{\eta}^*=0} ,$$

$$\langle\psi^*(x_1)\psi(x_2)\rangle = \{\delta/\delta\bar{\eta}^*(x_1)\}\{\delta/\delta\bar{\eta}(x_2)\} \langle 0|0\rangle|_{\bar{\eta}=\bar{\eta}^*=0}$$

Generally, if $f[\psi,\psi^*]$ is a functional of $\psi(x)$ and $\psi^*(x)$, its expectation value is given by

$$\langle f[\psi,\psi^*]\rangle = f[\delta/\delta\bar{\eta}, \delta/\delta\bar{\eta}^*] \langle 0|0\rangle|_{\bar{\eta}=\bar{\eta}^*=0} , \tag{4}$$

In order to obtain $\langle 0|0\rangle$ as a function of $\bar{\eta}$ and $\bar{\eta}^*$, we put

$$\langle 0|0\rangle = e^{\theta} , \tag{5}$$

and expand θ with respect to $\bar{\eta}$, $\bar{\eta}^*$, and also to the external sources η and η^*:

$$\begin{aligned}\theta = {} & \int dx dx' [\bar{\eta}(x)\kappa_{01}(x|x')\eta(x') + \bar{\eta}^*(x)\kappa_{10}(x|x')\eta^*(x')] \\ & + \frac{1}{(2!)^2} \int dx_1 dx_2 dx_1' dx_2' [\bar{\eta}(x_1)\,\bar{\eta}(x_2)\kappa_{02}(x_1,x_2|x_1',x_2')\eta(x_1')\eta(x_2') \\ & + 4\bar{\eta}^*(x_1)\bar{\eta}(x_2)\kappa_{11}(x_1;x_2|x_1';x_2')\eta^*(x_1')\eta(x_2') \\ & + \bar{\eta}^*(x_1)\bar{\eta}^*(x_2)\kappa_{20}(x_1,x_2|x_1',x_2')\eta^*(x_1')\eta^*(x_2')] + \dots \end{aligned} \tag{6}$$

$$\begin{aligned} = {} & \sum_{\nu,\mu=1}^{\infty} \frac{1}{(\nu!\mu!)^2} \int \prod_{i=1}^{\nu} dx_i dx_i' \prod_{j=1}^{\mu} dy_j dy'_j\, \bar{\eta}^*(x_1)\dots\bar{\eta}^*(x_\nu)\bar{\eta}(y_1) \\ & \times \dots\bar{\eta}(y_\mu)\, \kappa_{\nu\mu}(x_1,\dots,x_\nu;y_1,\dots,y_\mu|x_1',\dots,x_\nu';y_1',\dots,y'_\mu) \\ & \times \eta^*(x_1')\dots\eta^*(x'_\nu)\eta(y_1')\dots\eta(y'_\mu). \end{aligned}$$

Here, the expansion coefficients, i. e., the cumulants $\kappa_{\nu\mu}$'s, have the symmetry

$$\kappa^*_{\nu\mu}(x;y|x';y') = \kappa_{\nu\mu}(y;x|y';x'), \tag{7}$$

and they are also symmetrical with respect to the coordinates in each group of $\{x_i\}$, $\{y_i\}$, $\{x_i'\}$, and $\{y_i'\}$

The cumulants $\kappa_{\nu\mu}(x;y|x';y')$ are independent of the external sources η, η^* and also of $\bar{\eta}$ and $\bar{\eta}^*$, and the complete statistical informations of the wave can be obtained in terms of them.

2. Relations between the Cumulants and the Green's Functions

The cumulants $\kappa_{\nu\mu}$ can be given in terms of the statistical Green's functions defined as follows:

$$G_{01}(x|x') = \{\delta/\delta\eta(x')\} <\psi(x)> \Big|_{\eta=\eta^*=0},$$

$$G_{10}(x|x') = \{\delta/\delta\eta^*(x')\} <\psi^*(x)> \Big|_{\eta=\eta^*=0}, \tag{8}$$

$$G_{11}(x;y|x';y') = \{\delta/\delta\eta^*(x')\}\{\delta/\delta\eta(y')\} <\psi^*(x)\psi(y)> \Big|_{\eta=\eta^*=0},$$

$$G_{\nu\mu}(x_1,\dots,x_\nu;y_1,\dots,y_\mu|x_1',\dots,x'_\nu;y_1',\dots,y'_\mu)$$

$$= \prod_{i=1}^{\nu} \{\delta/\delta\eta^*(x'_i)\} \prod_{j=1}^{\mu} \{\delta/\delta\eta(y'_j)\} <\psi^*(x_1)\dots\psi^*(x_\nu)\psi(y_1)\dots$$

$$\dots \psi(y_\mu)>\Big|_{\eta=\eta^*=0}$$

$$= \prod_{i=1}^{\nu} \{\delta/\delta\bar{\eta}^*(x_i)\}\{\delta/\delta\eta^*(x'_i)\} \prod_{j=1}^{\mu} \{\delta/\delta\bar{\eta}(y_j)\}\{\delta/\delta\eta(y'_j)\}$$

$$\times <0|0> \Big|_{\eta=\eta^*=\bar{\eta}=\bar{\eta}^*=0},$$

where the last expression is obtained by the use of (4).

Thus, the Green's functions thus defined are the expansion coefficients of $<0|0>$ with respect to η, η^*, $\bar{\eta}$ and $\bar{\eta}^*$, and have the same symmetries as in (7).

Using the definition (6) of the cumulants, it is straightforward to obtain the following expressions of $\kappa_{\nu\mu}$ in terms of the Green's functions:

$$\kappa_{10}(x|x') = G_{10}(x|x'), \qquad \kappa_{01}(x|x') = G_{01}(x|x'),$$

$$\kappa_{11}(x;y|x';y') = G_{11}(x;y|x';y') - G_{10}(x|x')G_{01}(y|y'),$$

$$\kappa_{20}(x_1,x_2|x_1',x_2') = G_{20}(x_1,x_2|x_1',x_2')$$

$$- G_{10}(x_1|x_1')G_{10}(x_2|x_2') - G_{10}(x_1|x_2')G_{10}(x_2|x_1'), \text{ etc.} \tag{9}$$

It is also useful to express the Green's functions in terms of the $\kappa_{\nu\mu}$'s:

$$G_{11}(x;y|x';y') = \kappa_{10}(x|x')\kappa_{01}(y|y') + \kappa_{11}(x;y|x';y'),$$

$$G_{21}(x_1,x_2;y|x_1',x_2';y') = \kappa_{21}(x_1,x_2;y|x_1',x_2';y')$$

$$+ \kappa_{11}(x_2;y|x_2';y')\kappa_{10}(x_1|x_1') + \kappa_{11}(x_2;y|x_1';y')\kappa_{10}(x_1|x_2')$$

$$+ \kappa_{11}(x_1;y|x_1';y')\kappa_{10}(x_2|x_2') + \kappa_{11}(x_1;y|x_2';y')\kappa_{10}(x_2|x_1')$$

$$+ \kappa_{20}(x_1,x_2|x_1',x_2')\kappa_{01}(y|y')$$

$$+ \{\kappa_{10}(x_1|x_1')\kappa_{10}(x_2|x_2') + \kappa_{10}(x_1|x_2')\kappa_{10}(x_2|x_1')\}\kappa_{01}(y|y'),$$

$$\text{etc.} \tag{10}$$

The cumulants $\kappa_{\nu\mu}$ can be considered to be more basic statistical quantities than the Green's functions; in the perturbation theory, the Feynman diagram representing $\kappa_{\nu\mu}$ is a connected graph which does not contain any disconnected graph.

The cumulants and the Green's functions are connected also by the important relation of the following form:

$$\kappa_{11}(x;y|x';y') = \int dx''dy''dx'''dy'''G_{10}(x|x'')G_{01}(y|y'')$$

$$\times\ I_{11}(x'',y''|x''';y''')G_{11}(x''';y'''|x';y'), \qquad (11)$$

or, more generally,

$$\kappa_{\nu\mu}(x;y\ x';y') \equiv \kappa_{\nu\mu}(x_1,\ldots,x_\nu;y_1,\ldots,y_\mu\ x_1',\ldots,x'_\nu;y_1',\ldots,y'_\mu)$$

$$= \int dx''dy''dx'''dy''' \prod_{i=1}^{\nu}\prod_{j=1}^{\mu} G_{10}(x_i|x_i'')G_{01}(y_j|y_j'')$$

$$\times\ I_{\nu\mu}(x'',y''|x''';y''')G_{\nu\mu}(x''';y'''|x';y'). \qquad (12)$$

If the "interaction" functions $I_{\nu\mu}(x;y|x';y')$ are known, the cumulants or the Green's functions can be obtained in principle by solving eqn. (12) with the relation (9) or (10).

The form of the $I_{\nu\mu}$'s depends on the statistical properties of the randum function $q(x)$. A system of equations is obtained to find $I_{\nu\mu}$, assuming the Gaussian multivariate distribution of $q(x)$ with an arbitary correlation function $D(x - x') = \langle q(x)q(x')\rangle$.

3. Power Distribution Function

The probability density function $P(w)$ of the power $w(x) = \psi^*(x)\psi(x)$ at a particular point x is very important in practical problems, and it is given by the integral

$$P(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\ e^{i\lambda w}\ f(\lambda). \qquad (13)$$

Here, on using (4) with (5), the characteristic function $f(\lambda)$ is given by

$$f(\lambda) = \langle \exp\{-i\lambda\psi^*(x)\psi(x)\}\rangle$$
$$= \exp[-i\lambda\{\delta/\delta\bar{\eta}^*(x)\}\{\delta/\delta\bar{\eta}(x)\}]e^{\theta}\big|_{\bar{\eta}=\bar{\eta}^*=0}, \qquad (14)$$

and θ is given by the expansion (6). If, in this expansion, the higher terms of $\nu+\mu \geq 3$ can be neglected, as it is the case in many practical problems, the evaluation of (14) yields

$$f(\lambda) = \{(1+i\sigma\lambda)^2 + \rho^*\rho\lambda^2\}^{-1/2} \qquad (15)$$
$$\times \exp[-i\lambda\{(1+i\sigma\lambda)^2+\rho^*\rho\lambda^2\}^{-1}\{(1+i\sigma\lambda)|\langle\psi\rangle|^2-\frac{i}{2}(\langle\psi\rangle^2\rho^*+\langle\psi^*\rangle^2\rho)\lambda\}],$$

where

$$\sigma = \langle\psi^*\psi\rangle - \langle\psi^*\rangle\langle\psi\rangle, \qquad \rho = \langle\psi^2\rangle - \langle\psi\rangle^2. \qquad (16)$$

When the propagation distance is long enough so that $|\rho| \ll \sigma$, the evaluation of the integral of (13) shows that $P(w)$ reduces to the well-known distribution of the signal $\langle\psi\rangle$ plus the Gaussian noise:

$$P(w) = \sigma^{-1}e^{-\{w+|\langle\psi\rangle|^2\}/\sigma}I_0(2\sqrt{w}|\langle\psi\rangle|/\sigma), \quad w \geq 0. \qquad (17)$$

A brief survey of a quantum mechanical method to obtain $\langle 0|0\rangle$ of (3) is also presented as another method; it is an extension of the previous paper* (for the type of equations of the Schrödinger equation) to cover a general case where the stochastic change of $q(x)$ is not a Markov process.

*
K. Furutsu, 'Application of the method of quantum mechanics in the statistical theory of waves in a fluctuation medium', Phys. Rev., 168, pp. 167 - 179 (1968).